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A Survey of Development on Analysis Methods and the Theories of Shells

Ali Majidpourkhoei

Department of Civil Engineering, National University of Skills (NUS), Tehran, Iran

*Correspondence should be addressed to Ali Majidpourkhoei, Department of Civil Engineering, National University of Skills (NUS), Tehran, Iran. Tel: (+88) ******; Email: amajidpour@tvu.ac.ir

ABSTRACT

Shells are the most widely used structural elements forming roofs of buildings and other structures for supporting loads to its surface. There are many types of shell forms that can be made and practical. The most common shell form is a cylindrical shell. In general, it can be said that the shell is a curved surface whose thickness is insignificant compared to its radius and other dimensions. There are various theories for the analysis of shell structures. How to apply each of these theories depends on the geometric shape, shell material, shell application and boundary conditions as well as applied loads. Researchers have searched for information about the analysis methods of shell structures using various methods. Some used experimental methods for shell structures while others used analytical methods. This is an evaluation of these methods. In this paper we will research some historical aspects of development of the theory of shells as well as their recent developments. A survey and introducing of the books and articles of methods of shells analysis have been discussed in this article.

Keywords: Theory of Shells, Shells Analysis, Shell Structures

1. INTRODUCTION

This article will focus mainly on shell theories. The discussion in this paper is intended to survey the historical development of the subject at hand. Bernoulli developed the first accurate equation for beams as early as 1735. His equations led to interesting discussion with Euler and solutions to several beam boundary conditions. The equations developed by Bernoulli assume pure bending, and thus. Both axial and shear deformations were ignored. It is now known that such equations are valid for thin beams undergoing small deformations. It was noted as early as 1877 that rotary inertia terms are important in the analysis of vibrating systems by Rayleigh (1877). More than 40 years later, Timoshenko (1921) showed that shear deformation terms are at least as important. The first accrete treatment of plates can be attributed to Germain (1821) and Lagrange (1811) early in the 19th century [1]. A good historical review of the development can be referred to in the books of Soedel (1993) and Timoshenko (1983) [2]. This theory is now referred to as the classical plate theory (CPT). It is valid for small deformation of thin plates. The inclusion of shear deformation in the fundamental equations of plates is due to Reissner (1945) [3]. Theories that account for shear deformation are now referred to as thick plate theories or shear deformation plate theories (SDPT). The first accurate cylindrical shell theory may be attributed to Love (1892) [4]. In this theory, Love introduced his first approximation for bending analysis of cylindrical shells. This approximation defines a linear analysis of thin shells, in which various assumptions were introduced. Among these assumptions, strains and displacements are assumed to be small such that second and higher order terms can be neglected. In addition, Love assumed the thickness of the shell to be small compared with other shell parameters, the transverse stress to be small compared with other stresses in shells, and normal to the unreformed surface to remain straight and normal's to the deformed surface. Since then, other shell theories were introduced that were based on the same Love's approximation but differed in the detailed derivation. Since the introduction of these shell theories, inconsistencies appeared in many of them. Leissa (1973) reported some of these inconsistencies with regard to rigid body motion and asymmetric differential operators [5]. To overcome some or all of these inconsistencies various theories were introduced including that of Sanders (1959) and Vlasov (1949) [6-7]. Among the additional developments that should be stated are the work of Reissner (1941) [3], Novozhilov (1958) [8], Timoshenko and Woinowsky (1959) [9], FIugge (1962), Donnell (1976) [10] and Mushtari (1961). Review of these developments and theories can be found in the monograph by Leissa (1973) [5] and the books by Kraus (1967) [11] and Soedel (1993). Since then, researchers realized that for thick beams, plates or shells both rotary inertia and shear deformation have to be included in any reliable theory of such components. Various studies, however, like that of Koiter (1967) [12] and Goldenveizer (1961) [13], concluded that for thin and moderately thick shells, the transverse normal stress remains small compared with other stresses in the cylindrical shell. The inclusion of shear deformation (and rotary inertia) led to a necessary relaxation of some of the assumptions in Love's first approximation, and shear deformation shell theories were born. Among the first of such theories were those of Vlasov (1949) [7], Reissner (1941) [3] and others. Cylindrical shells are three dimensional (3D) bodies bounded by two, relatively close, curved surfaces. The 3D equations of elasticity are complicated when written in curvilinear, or cylindrical shell, coordinates. Most engineers who dealt with shells tried to simplify such cylindrical shell equations by making certain assumptions for particular applications. Almost all cylindrical shell theories (thin and thick, deep and shallow, etc.) reduce the 3D elasticity problem into a 2D one. This is done usually by eliminating the coordinate normal to the cylindrical shell surface in the development of the cylindrical shell equations. The accuracy of thin and thick cylindrical shell theories can only be established if the results obtained by these theories are compared with those obtained using the 3D theory of elasticity. The development cylindrical shell equations matured over the last century. The expansion of the cylindrical shell equations to laminated composites occurred mostly in the second half of the 20th century by mainly a group of Russian researchers. The method that is followed in this paper is to start with the 3D elasticity theory in curvilinear coordinates. This is followed by introducing the necessary assumptions to reduce the 3D elasticity equations to those which take simpler forms and are applicable to beams, plates and shells. The equations described for cylindrical shells using curvilinear coordinates show the level of complexity that can be reached when treating laminated composite structures. This complexity in the equations of motion and the associated boundary conditions resulted only in a limited number of problems where exact solutions are possible. In general, however, it is known that there is no exact solution for a laminated structure (solid, plate or shell) with general boundary conditions and/or lamination sequence. Selected books and articles about the theory of plates and shells are given in the references [14-44].

2. RECENT DEVELOPMENTS OF 3D ELASTICITY THEORY

A shell is a 3D body confined by two surfaces. If these surfaces are parallel, the shell will have a constant thickness. In general, the distance between those surfaces is small compared with other shell parameters like length, width and radii of curvature. In this section, recent developments of the 3D theory of elasticity in curvilinear coordinates are presented. The recent research that used the 3D elasticity theory in the analysis of laminates shell structures is covered in a recent book [45]. In addition, the following work should be reported. Bhimaraddi (1991) obtained results based on the 3D theory of et elasticity for doubly curved composite shallow shells, Others, including Wang et al. (1995), Jiang (1997) and Tsai (1991), obtained results based on, the 3D theory of elasticity for closed cylindrical shells. 3D solutions for cylindrical shells with initial stresses are found by Xu et al. (1997). Ye and Soldatos (1996, 1997) used 3D elasticity theory to treat cylindrical shells with arbitrary point supports and clamped edge boundaries, respectively. Chen and Shen (1998) used 3D analysis to study orthotropic piezoelectric circular cylindrical shells. Chen et al. (1998) presented 3D study for free vibration of transversely isotropic cylindrical panels. Ding and Tang (1999) studied 3D free vibration of thick laminated cylindrical shells with clamped edges. Yin (1999) found simplifications for the frequency equation of multilayered cylinders and developed some recursion formulae of Bessel functions. Chern and Chao (2000) used 3D theory to study the natural frequencies of laminated curved panels. A survey of the 3D literature can be found in Soldatos (1994).

3. THICK SHELL THEORY

The first reliable approximation for bending analysis of shells was introduced by Love (1892) [4]. Love made several assumptions to reduce the general 3D equation of elasticity in curvilinear coordinates to 2D equations that can be applied for shells. First, he assumed that strains and displacements are small such that second and higher order terms can be neglected. Love's second assumption was that the thickness of the shell is small compared with other shell parameters. His third assumption was that the transverse stress is small compared with other stresses in shells. Finally, Love assumed that normal's to the unreformed surface remains straight and normal to the deformed surface. This assumption needs to be relaxed if the stains and/or displacements become large. Displacement is considered definitely large if it exceeds the thickness of the cylindrical shell. This is typical for thin shells. Nonlinear behavior can be observed even before this level of deformation for various boundary conditions. A recent study concluded that this assumption generally applies to most of the analyses of thick shells [46]. This is because stresses exceed allowable values before the deflection becomes large enough for the nonlinear terms are important. The remaining assumptions in Love's first approximation need to be reexamined when thick shells are treated. For thick shells, the thickness is no longer small compared with other shells parameters, nor do the normals of the unreformed surface remain as such. Various studies concluded that even for thicker shells the transverse normal stress (and strain) remains small compared with other stresses (and strains) in the shell [48]. As pointed earlier, many shell theories were derived based on Love's first approximation. Inconsistencies, however, appeared in many of these theories and were reported in much of the literature on shell theory during the middle of the 20th century [5]. For example, the strain-displacement relations used by Naghdi and Berry (1964) are inconsistent with regard to rigid body motion. Other theories including love (1892) [4] and Timoshenko and Woinowsky (1959) [9], although free from rigid body motion inconsistencies, introduced unsymmetrical differential operators, which contradicts the theorem of reciprocity and yields imaginary numbers for natural frequencies in a free vibration analysis. Other inconsistencies appeared when the assumption of small, thickness (h/R and z/R<1) is imposed and symmetric stress resultants ($N_{\alpha\beta} = N_{\beta\alpha}$ and $M_{\alpha\beta} = M_{\beta\alpha}$) are obtained. This is not true for shells that are not spherical. To overcome some or all the above inconsistencies, various theories were introduced including that of Sanders (1959) [49]. Vlasov (1949) [7] tried to resolve some of these inconsistencies by expanding usually negligible terms appearing in the denominator of the stress resultant equations using a Taylor series. On the other hand, Rayleigh noted the importance of rotary inertia terms in the analysis of vibrating systems. Timoshenko (1921) showed that shear deformation terms are at least as important. This led to a necessary relaxation of some of the assumptions in

Love's first approximation and shear deformation shell theories (SDST) were born. Among the first of such theories were those of Vlasov (1949) [7], Reissner (1941, 1952) [3], Naghdi and Cooper (1956) and others. Ambartsumian (1961) [50] expanded the stress resultant equations of earlier theories to those for anisotropic shells. Orthotropic cylindrical shells were considered by Dong (1968) and Rath and Das (1973). The latter presented equations that included rotary inertia and shear deformation. Various survey articles appeared on the treatment of homogeneous and laminated composite shells. These articles reviewed theories and analyses of laminated composite shells. It was found that shear deformation effects for laminated composite materials are generally more important than those for isotropic materials. Unfortunately, while SDST, including higher order ones, include shear deformation and rotary inertia, they fail to consider the 1+ z/ R terms in the stress resultant equations. This led to inaccurate results in the constitutive equations used for laminated deep thick shells. This was initially observed by Bert (1967) and investigated thoroughly only recently by Qatu (1993) for curved beams and Leissa and Chang (1996) [31] and Qatu (1995, 1999) for laminated shells. Leissa and Chang truncated this term using a geometric series expansion. Qatu integrated the term exactly. Results obtained using this theory show closer comparison with 3D theory of elasticity. Other shear deformation theories included higher order terms for shear strains, but neglected the 1 + z/R [51-52]. Interestingly, the results obtained by these authors were presented for shallow shells. For such shells the term (1 + z/R) is less important. Assuming that normal's to the mid surface strains remain straight during deformation but not normal to the mid surface, the displacements can be written as:

$$
\begin{aligned}
u(\alpha,\beta,z) &= u_o(\alpha,\beta) + z\psi_\alpha(\alpha,\beta)\\
v(\alpha,\beta,z) &= v_o(\alpha,\beta) + z\psi_\beta(\alpha,\beta)\\
w(\alpha,\beta,z) &= w_o(\alpha,\beta)
\end{aligned}
\qquad (1)
$$

Where uo, vo and wo are mid surface displacements of the shell and ψ_α & ψ_β are mid surface rotations. The third of the above equations yields $\varepsilon_z = o$. An alternative derivation can be made with the assumption $\sigma_z = \upsilon$ [52]. The subscript (0) will refer to the middle surface in the subsequent equations. The above equations describe a typical first order SDST and will constitute the only assumption made in this theory when compared with the 3D theory of elasticity described earlier.

3.1. Recent development of thick shell theory

Many recent research papers appeared where thick, or shear deformation, shell theories have been used. Such theories were used by Kabir (1998), Kabir and Chaudhuri (1991, 1994), Nosier and Reddy (1992), Piskunov et al. (1994, 1995), Soldatos (1991), Wang and Schweizerhof (1996, 1997), Sivadas (1995) Sharma et al. (1999), Xi et al. (1996), Argyris and Tenek (1996), Chonan, (1988), Ding et al. (1997) and Toorani and Lakis (2000).

Higher order shear deformation theories were developed and used for composite shells by Librescu et al. (1989), Mizusawa (1996), Ma and He (1998), Messina and Soldatos (1999) and Timarci and Soldatos (1995). Thick circular cylindrical shells were considered by Birlik and Mengi (1989), Sivadas and Gauesan (1991, 1993), Kolesnikov (1996), Sun et al. (1997), Zellkour (1998), Soldatos and Messina (1998) and Mizusawa (1996). Non-circular cylindrical shells were reviewed by Soldatos (1999). Spherical thick shells were studied by several researchers (Chao and Chern, 1988; Dasgupta and Huang, 1997; Gautham and Ganesan, 1994), Lu et al. (1996), Ramesh and Ganesan (1993) and Sivadas and Ganesan (1992) considered thick conical shells. The second approach to the development of shell theories is a stress-based approach. The work of Reissner is the first to adopt this approach, which has recently been used by Yong and Cho (1995), Carrera (1999) and Xi et al. (1996). This will not be described here.

4. THIN SHELL THEORY

If the shell thickness is less than 1/20 of the wavelength of the deformation mode and / or radii of curvature, a thin shell theory, where both shear deformation and rotary inertia are negligible, is generally acceptable. Some researchers question the value of 1/20 and choose a smaller value, particularly for laminated composite materials. Depending on various assumptions made during the derivation of the strain-displacement relations, stress-strain relations, and the equilibrium equations, various thin shell theories can be derived. Among the most common of these are the Love, Reissner, Naghdi, Sander and Flugge shell theories. All these theories were initially derived for isotropic shells and expanded later for laminated composite shells by applying the appropriate integration through lamina and stress-strain relations. Leissa (1973) showed that most thin cylindrical shell theories yield similar results [5]. The exceptions were the membrane theory and that of Donnell and Mushtari. The equations of Donnell and Mushtari describe shallow shells and are shown later as a part of shallow, cylindrical shell theory .The equations shown in this section are equivalent to those of Reissner and Naghdi with the extension to laminated composite thin shells. The reason for adopting these equations is the fact that they offer the simplest, most accurate and consistent equations for laminated thin cylindrical shells.

4.1. Recent development of thin shell theory

We will describe some of the recent literature in which thin shell theories were used to analyze composites. Love's shell theory was used by Sivadas and Ganesan (1992), Ip et al. (1996), Shu (1996) and Lam and Loy (1995). Sanders' theory was employed by Selmane and Lakis (1997), Lam and Loy (1995), Mohd and Dawe (1993) and Bhattacharyya and Vendhan (1991). Flugge's theory was used by Narita et al (1993). Loy et al. (1999) studied cylindrical panels with different boundary conditions. Recently, Qatu (1999), presented a theory for isotropic and laminated barrel thin shells, and used it to study vibrations.

5. LAYER-WISE THEORIES

Other thick shell theories exist. Among these are layer-wise theories. In these theories, each layer in a laminated shell is considered to be a shell, and compatibility conditions are applied between various layers. Layer-wise shell theories have been used by Huang and Dasgupta (1995), Dasgupta and Huang (1997) and Gautham and Ganesan (1994) for free vibrations of thick composite cylindrical and spherica1panels. Carrera used layer-wise theories to evaluate doubly curved shells. Xavier (1995) used one to study the effect of various curvatures. Basar and Omurtag (2000) examined vibrations of laminated structures using layer–wise shell models.

6. NONLINEAR THEORIES

The magnitude of transverse displacement compared to cylindrical shell thickness is another criterion used in classifying shell equations. It can be shown that if such magnitude approaches the thickness of the shell (often earlier than that), the results of a linear analysis can be in gross error. The nonlinear terms were neglected in the thin; thick and 3D shell theories described earlier. In nonlinear shell theories, such terms are retained. In many cases, they are expanded using perturbation methods, and smaller orders of the protestations are retained. Most frequently the first order only is retained and, recently, even third orders have been included in a nonlinear shell theory by Pai and Nayfeh (1992-1994) [54]. In some shell dynamics problems, the material used can also be nonlinear (rubber, plastics and others). Theories that include nonlinear material behavior are also referred to as nonlinear shell theories. The vast majority of shell theories, however, deal with geometric nonlinearity only. A nonlinear thin shell theory was used to study open cylindrical shells by Selmane and Lakis (1997) and closed ones by Li (1996) and Ganapathi and Varadan (1996). A nonlinear shallow thin shell theory was' used by Raouf and Palazotto (1994), Li (1993) and Xu and Chia (1994). Doubly curved shallow shells having simple supports were treated by Shin (1997) using a nonlinear shell theory. Shallow shells of revolution including spherical and conical shells were also studied by Li (1992). Effects of shear nonlinearity on the free vibration of laminated shells were discussed by Xi (1999). In various research articles both shear deformation and nonlinear terms were included (Reddy 1989)

[55]. It is argued that such theories will solve only a limited number of problems and are not needed for general composites. Large deformation of open, doubly curved shallow shells was treated by Tsai and Palazotto (1991), Noor et al. (1991, 1994) and recently by Shin (1997) and Wang et al. (1997). Xu et. al. (1996) studied truncated thick shallow shells. Shallow spherical shells were considered by Xu and Chia (1994) and Sathyamoorthy (1995). Tang and Chen (1998) studied the nonlinear analysis of laminated composite cylindrical panels. Fu and Chia (1989, 1993), Ganapathi and Varadan, (1995, 1996), Soldatos (1992), Cederbaum (1992) and lu and Chia (1988) studied thick circular cylindrical shells. Selmane and Lakis (1997) and Lakis et al. (1998) studied the influence of geometric nonlinearities on the free vibrations of orthotropic open and closed cylindrical shells. Roussos and Mason (1998) studied the nonlinear radial vibrations of a hyper elastic cylindrical tube. Pai and Nayfeh (1992, 1994) [54] presented equations that included higher order shear deformation terms as well as higher order nonlinearities for composite cylindrical shell. Gummadi and Palazotto (1999) used finite elements to analyze nonlinear dynamics of composite cylindrical shells considering large rotations. Conical shells were considered by Xu et al. (1996). Zarutski (1998) developed approximate nonlinear equations of motion of cylindrical shells. Abe et al. (2000) studied the nonlinear vibration of clamped composite shallow shells. Cho et al. (2000) studied nonlinear behavior of composite shells under impact using finite elements. Other studies on nonlinear dynamics of composite shells include Nayfeh and Rivieccio (2000).

7. EXPRIMENTAL METHODS

Experimental results for laminated composite shells were obtained by a limited number of studies. Some of these studies included using scaled down models and similitude theory. The use of similitude was explained for isotropic shells by Soedel (1993). Influence of enclosed air. On vibration modes of a shell Structure was discussed by Isaksson et al. (1995). Experiments show that enclosed air in a thin walled structure affects some modes of vibration significantly. Air coupling between vibrating sides of the structure cannot always be neglected, and frequencies may not be predicted accurately if calculations are performed as if in a vacuum. Optical modal analysis of the real, physical, violin model has been performed by using electronic holography. Calculated modes of vibration were compared with experimental ones. For the lowest modes, a good agreement between measured and calculated frequencies was reached. A mixed numerical and experimental approach for the dynamic modeling of a composite shell was discussed by Swider et al. (1996). More discussion of experimental analysis of shell dynamics can be found in the article by Noor et al. (1996) and Qatu (2002) [47].

7.1. Exact solutions

The term "exact solutions" is used here to mean finding a solution that satisfies both the differential equations and boundary conditions exactly. It will not mean finding an exact solution to the 3D elasticity equations. Instead, the exact solutions will be searched for and presented only for the various theories that were described and will be given in further details for more specific geometries like curved beams, plates and shallow and cylindrical shells. Finding an exact solution to the set of shell equations and boundary conditions for a general lamination sequence and/or boundary conditions is out of reach. Only a limited number of boundary conditions and lamination sequences have exact solutions. Generally speaking, untwisted ($R_{\alpha\beta} = \infty$), symmetric and unsymmetrical cross-ply (0°, 90°) laminated singly or doubly curved shells (and plates) with two opposite edges having shear diaphragm boundaries permit exact solutions [52]. Exact solutions can also be found for plates with ant symmetric (-45°, 45°) lamination sequence with S3 boundaries at opposite edges. Such solution cannot be expanded for shells. Details of these solutions will be discussed later. The complexity of the analysis and solution available for such shells with two opposite edges simply supported, while the others are arbitrary, prevented many researchers from getting results. For closed shells, cylindrical or barrel, and open doubly curved shells or flat plates having all four edges with shear diaphragms, the solution becomes relatively simple. That is the reason for using such a solution in many publications on the subject. 3D solutions for cylinders were presented by Soldatos (1994). Recently, laminated cross-ply, thin, noncircular cylindrical shells were analyzed exactly for their natural frequencies by Suzuki et al. (1994). Laminated thick noncircular cylindrical shells were analyzed exactly as well by Suzuki et al. (1996). Exact solutions will be discussed later for the specific structural elements discussed here (curved beams, plates or shells). On the other hand, numerous approximation methods are available for researchers to study shell dynamics and obtain natural frequencies and mode shapes. The most used analytical methods will be the emphasis of this section. The various experimental methods used by researchers to obtain the needed information for laminated shell and plate vibrations will not be treated in this paper.

7.2. Approximate solutions

The methods described here can be used to obtain approximate solutions. The methods fundamentally assume a solution with undetermined coefficients. These coefficients are minimized to achieve a stationary value for certain functional or minimize errors. In the Ritz analysis this functional is the energy functional [1]. In other methods like the weighted residuals methods, this functional is some expression of deviation from the solution. The assumed functions can be continuous function describing the whole structure of a linear combination of various functions. Unless the assumed functions are actually the exact solution itself, which is rarely the case, errors are introduced. As the number of terms, or functions, increases, error is reduced, and convergence is observed. An issue that is treated considerably in the literature about these methods is to find whether these solutions converge to the exact solution or to some other solution within the vicinity of the exact solution. It is worth mentioning here that the analytical models described earlier, including the 3D elasticity model, are themselves approximate. The shell theories derived earlier are based on, and an approximation of, the 3D theory of elasticity. Thus, the accuracy of these theories in analyzing the actual physical problem is less than that of the 3D theory of elasticity. Approximate solutions introduce less accurate results when compared with exact solution, unless proper convergence characteristics are established. The variation or energy methods used for vibration analysis of continuous systems are the Ritz, and Ritz-based finite element methods (FEM). The most widely used weighted residuals methods are the Galerkin method and Galerkin based FEM. The FEM are piecewise applications of Galerkin and Ritz methods. Finite elements, however, is the dominant method used by application engineers to study the vibration behavior of structures including composite ones. Other variation methods are also used to a lesser degree by researchers in the field. Most of these are other weighted residual methods including Treffetz, collocation and point matching. There are other approximate methods used in the literature. One method that was used frequently by researchers is the finite difference method. This method approximates the derivatives in the governing differential equations by difference equations. While this method received considerable attention decades ago [11], it is receiving less attention in recent literature. The reader is encouraged to consult other texts for a comprehensive review of these methods [51].

8. THE RITZ METHOD

The Rayleigh and Ritz methods are among the most common approximate methods used in the vibration analysis of continuous systems. A displacement field is assumed in both methods. The coefficients of the displacement field are completely determined beforehand in the method of Rayleigh. In the Ritz method, undetermined coefficients are used in the displacement field. The displacement field is then substituted in the energy functional. The Lagrangian is then minimized by taking its derivatives with respect to these coefficients and making them equal to zero. This yields equations that can be written in a matrix form and give the displacement in a static analysis and natural frequencies in a vibration analysis. The method of Rayleigh, which assumes a completely determined mode shape, generally yields a less accurate frequency corresponding to the assumed mode shape when compared with the Ritz method. For free vibration analysis, the motion is assumed to be harmonic and the displacements take the form:

$$u(\varphi_1,\varphi_2,\varphi_3,t)=U(\varphi_1,\varphi_2,\varphi_3)\sin\omega t$$
$$v(\varphi_1,\varphi_2,\varphi_3,t)=V(\varphi_1,\varphi_2,\varphi_3)\sin\omega t$$
$$w(\varphi_1,\varphi_2,\varphi_3,t)=W(\varphi_1,\varphi_2,\varphi_3)\sin\omega t \quad (2)$$

Where φ_1,φ_2 and φ_3 are the coordinates used in the problem and t is time. In the Ritz method, displacement functions U, V and W are approximated by using certain functions or a linear combination of such functions. For example, the displacements in a 3D problem can be approximated by using:

$$U(\varphi_1,\varphi_2,\varphi_3)=\sum_{i=0}^{I}\sum_{j=0}^{j}\sum_{k=0}^{K}\alpha_{ijk}f_{ijk}(\varphi_1,\varphi_2,\varphi_3)$$
$$V(\varphi_1,\varphi_2,\varphi_3)=\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N}\beta_{lmn}f_{lmn}(\varphi_1,\varphi_2,\varphi_3)$$
$$W(\varphi_1,\varphi_2,\varphi_3)=\sum_{p=0}^{P}\sum_{q=0}^{Q}\sum_{r=0}^{R}\gamma_{pqr}f_{pqr}(\varphi_1,\varphi_2,\varphi_3) \quad (3)$$

Where α_{ijk},β_{lmn} and γ_{pqr} are the undetermined coefficients and f_{ijk} , f_{lmn} and f_{pqr} are the assumed functions. These functions will need to be determined first. They must satisfy two conditions for the solution to be of any value. First, they should be continuous (for a continuous structure), linearly independent and differentiable to the degree needed in the problem. Second, they should satisfy at least the geometric boundary conditions. They do not need to satisfy the forced boundary conditions [55]. Numerical investigations, however, showed that when these assumed displacements satisfy the forced boundary conditions, a more rapid convergence would be observed. In order to guarantee convergence to the exact solutions, an additional condition must be met. This condition is that the series of these functions (frequently referred to as trial functions or shape functions) need to be mathematically complete. Mathematical completeness of a series of functions will mean that any function (actual displacements) can be adequately represented by these complete functions. Completeness has been proven for many series of functions [56]. These include both algebraic polynomials and trigonometric functions. Both are widely used with the Ritz method to obtain natural frequencies. The method of least squares in curve fitting proves that practically any function can be represented by an infinite series of algebraic polynomials [56]. Fourier series expansion proves the same for trigonometric series of functions. For a detailed discussion on completeness of functions, one can consult the books by Kantorovich and Krylov (1964) and Kreyszig (1989, 1993) [57].

$$U(\varphi_1,\varphi_2,\varphi_3)=\sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K}\alpha_{ijk}\varphi_1^i\varphi_2^j\varphi_3^k$$
$$V(\varphi_1,\varphi_2,\varphi_3)=\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N}\beta_{lmn}\varphi_1^l\varphi_2^m\varphi_3^n$$
$$W(\varphi_1,\varphi_2,\varphi_3)=\sum_{p=0}^{P}\sum_{q=0}^{Q}\sum_{r=0}^{R}\gamma_{pqr}\varphi_1^p\varphi_2^q\varphi_3^r \quad (4)$$

The first equation in Eq. 3 can be expanded as:

$$U(\varphi_1,\varphi_2,\varphi_3)=\sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K}\alpha_{ijk}\varphi_1^i\varphi_2^j\varphi_3^k=$$
$$=\alpha_{000}+\alpha_{001}\varphi_3+\alpha_{010}\varphi_2+\alpha_{100}\varphi_1+$$
$$+\alpha_{111}\varphi_1\varphi_2\varphi_3+\alpha_{112}\varphi_1\varphi_2\varphi_3^2+$$
$$+\alpha_{113}\varphi_1\varphi_2\varphi_3^3+\ldots+\alpha_{11k}\varphi_1\varphi_2\varphi_3^k+$$
$$+\alpha_{121}\varphi_1\varphi_2^2\varphi_3+\alpha_{131}\varphi_1\varphi_2^3\varphi_3+$$
$$+\ldots+\alpha_{1J1}\varphi_1\varphi_2^J\varphi_3+\alpha_{211}\varphi_1^2\varphi_2\varphi_3+$$
$$+\alpha_{311}\varphi_1^3\varphi_2\varphi_3+\ldots$$
$$+\alpha_{I11}\varphi_1^I\varphi_2\varphi_3+\alpha_{222}\varphi_1^2\varphi_2^2\varphi_3^2+\ldots \quad (5)$$

It should be noted here that no higher order terry should be used unless the needed lower order terms in used for the same power series. The needed lower order terms are all the lower order terms that do not violate the necessary geometric boundary conditions. Once the displacement field or assumed functions are determined, they need to be directly substituted in the Lagrangian energy functional

L=T -U+W. The total expression of the Lagrangian needs to be minimized in search of a stationary value. This search requires minimization of the expression with respect to the undetermined coefficients:

$$\frac{\partial L}{\partial \alpha_{ijk}} = 0, \quad i = 0,1,\ldots,I, \quad j = 0,1,\ldots,J, \quad k = 0,1,\ldots,K,$$
$$\frac{\partial L}{\partial \beta_{lmn}} = 0, \quad l = 0,1,\ldots,L, \quad m = 0,1,\ldots,M, \quad n = 0,1,\ldots,N,$$
$$\frac{\partial L}{\partial \gamma_{pqr}} = 0, \quad p = 0,1,\ldots,P, \quad q = 0,1,\ldots,Q, \quad r = 0,1,\ldots,R, \qquad (6)$$

In a typical problem with no geometric constraints, this yields a system of equations of the size (I + 1)(J + 1)(K + I) + (l+1)(M + 1)(N + 1) + (P + 1)(Q + 1)(R + 1). In order to avoid the trivial zero solution, the determinant must be taken to zero. In a free vibration analysis, this yields a set of Eigen values. Furthermore, generally speaking, the force resultants and stresses obtained by using the Ritz method will be less accurate when compared with displacements and frequencies. The assumed displacements in a Ritz analysis converge to the exact frequencies only if a complete set of assumed functions is used. Since that is not the case in typica1 problems where the series get truncated at some level, the assumed displacement field usually represents a stiffer system than the actual one. This results in higher frequencies than the exact solution. This is established numerically by observing the convergence characteristics of the natural frequencies. These frequencies converge monotonically from above in a Ritz analysis. The Ritz method has been widely used in mechanics for a variety of problems, most notably in vibrations of continuous systems. The method can be used in nonlinear vibration analysis, resulting in a nonlinear system of algebraic equations. These equations are then solved using numerical techniques leading to more than one answer.

The method of Ritz permits obtaining as many natural frequencies as needed. If a complete set of functions is used, the Ritz method has excellent convergence characteristics and is relatively easy to program [53]. The Ritz method, sometimes referred to as Rayleigh and Ritz, has been used successfully to analyze thin cylindrical shells. It has also been used to study thin shallow shells. The Ritz method has also been used to study thick shallow shells by Singh and Kumar (1996) and spherical shells by Chao and Chern (1988), and other shells by Lee (1988). Raouf and Palazotto (1992) used the Ritz method with harmonic balance to analyze nonlinear vibrations of shells. One has to start with a displacement field that satisfies at least the geometric boundary conditions when using the Ritz method. Furthermore, it is somewhat difficult to obtain a displacement field for a relatively complex shell structure or set of such structures. For example, a problem of a plate attached to a shell at a certain angle is challenging if the Ritz method is to be used, as found by Young and Dickinson (l997). Special treatments were introduced to overcome the difficulty of using the Ritz method with general boundary conditions.

9. THE GALERKIN METHOD

The Ritz method is used to obtain approximate solution to problems having an energy expression. There are problems in mechanics which do not allow for such an expression. The weighted residual methods are basically a generalization of the Ritz method to handle problems that do not allow an energy expression. The Galerkin method is a special case of the general weighted residual methods. In these methods, the governing differential equations and corresponding boundary conditions are needed. For a typical problem treated in this text, the governing equations can be written in the form:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad (7)$$

Where the L_{ij} terms are the differential operators (which may contain derivatives with respect to time), U, V and where the displacement fields and p1, p2 and p3 are the forcing function. Additional boundary terms will represent specified boundary conditions. The solution of the displacement field using the Galerkin method is searched for in a form similar to Eq. 2 for a free vibration analysis. The displacement field is approximated with trial functions, Similar to Eq. 3. This displacement field is displacement field is directly substituted in the governing differential Eq. 7. This yields an expression for the residual that can be written as:

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \times$$
$$\times \begin{bmatrix} \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \alpha_{ijk} f_{ijk}(\varphi_1,\varphi_2,\varphi_3) \\ \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{n=0}^{N} \beta_{lmn} f_{lmn}(\varphi_1,\varphi_2,\varphi_3) \\ \sum_{p=0}^{P} \sum_{q=0}^{Q} \sum_{r=0}^{R} \gamma_{pqr} f_{pqr}(\varphi_1,\varphi_2,\varphi_3) \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad (8)$$

They are functions of the undetermined coefficients, the forcing functions and the coordinate system. The weighted residual methods require these coefficients to be orthogonal to some weight functions ψ_i.

$$\int_V \psi_i E_n dV = 0, \quad i = 1,2,\ldots,n \quad (8)$$

Various weighted residual methods differ from each other in the selection of the weight functions. Galerkin's method requires those functions to be the trial functions themselves. Interestingly, for many problems this yields computations similar to those of the Ritz method.

The Galerkin method has been used mainly to study nonlinear vibration problems. It, was used to analyze doubly curved shallow shells in many research articles. The Galerkin method was also used to study the nonlinear vibration of cylindrical shells in various studies. Other researchers used the method to deal with conical shells. Spherical shells were studies by Chao et al. (1991), Narasimhan and Alwar (1992), and Xu and Chia (1994, 1995). Other shells were considered by Fu and Chia (1989).

10. THE FINITE ELEMENT METHODS

The FEM have been growing rapidly in the last three decades. Most of these methods have been based upon the Ritz method (minimizing the energy functional) or other weighted residual methods including Galerkin on an element level to obtain an element stiffness matrix. FEM overcome the difficulties the Ritz and Galerkin methods have in dealing with various boundary conditions and relatively complex shapes. For simple structures, the Ritz method shows better convergence and less computational needs. For complex structures, loading functions and/or boundary conditions, the FEM have proven to be an excellent tool. A considerable number of FEM commercial codes and packages exist to obtain vibration results. Good literature survey articles on the subject were presented by Noor and Burton (1992), Noor et al. (1993) and Reddy (1989) [55]. The main difference the finite element methods have from the previous methods is that the equations are derived at the sub domain (called finite element) level. The finite elements themselves are simple in shape. Algebraic polynomials are typically used as the trial functions at the element level which makes the computation relatively straightforward. The undetermined coefficients are the values of dependent variables at certain selected points within the structure (called nodes). Finite element analysis will be described briefly here. For a detailed derivation, the reader is advised to consult one of the major texts on the subject. In today's commercial finite element packages, the actual computations are transparent to the user. What the user actually encounters are three fundamental components of the finite element package. The first is the preprocessor. In this component, a finite element model is built; this modeling part constitutes the major task done by today's finite element engineers. In this task, the structure at hand is divided into elements. The user must decide whether solid elements, shell elements or other elements are to be used. There are some general guidelines on the applicability and usage of each of these elements. These guidelines are driven by the accuracy needed, the complexity of the structure at hand, and the computational time available. Generally speaking, smaller element sizes (larger number of elements for the whole structure) will need more computational time and deliver more accurate results. If shell elements are used, only the middle surface needs to be defined in the model and the thickness is presented as an element property. As a general rule, the element side lengths used when modeling a structure using shell elements should be larger than the thickness by one order of magnitude. Otherwise, solid elements should be selected. Also, if solid elements are used, there should be at least two, preferably four, elements across the thickness, once the general type of elements is decided (solid versus shell), the type of element should be further specified. There are many different solid and shell elements available to the user. For solid elements, the user can use tetrahedron elements, parallel-piped ones (referred to as brick elements) or others. Each type of these has its own advantages and shortcomings. For shell elements, the user often has to choose triangular, rectangular or other element. For each of these, there are multiple derivatives. For example, there are four-node rectangular elements, eight-node ones and others. Again, advantages and shortcomings are reported for each. The user can also mix elements relatively easily when elements are chosen from the same category (solids) and with great care when elements from different categories are connected together (connecting solid elements with shell elements). The mismatch of the degrees of freedom (DOF) at the nodes when shell elements are connected to solid elements should be treated very carefully. Figure (1) display a shell modeled by shell elements, respectively.

Once the model is built, most commercial packages have a variety of tools to check the model. Among the most important checks is the inner connectivity of the elements. This is important to avoid unintended internal boundaries between the elements. For shell elements, the normals of the elements should be in the same direction. Processor code reads the input data prepared by the preprocessor. This code contains material properties of each element, nodal coordinate data as well as the nodes surrounding each element. The code is usually editable by the user. The processor itself is usually transparent to the user and is the portion that requires considerable computational timing depending on the refinement of the model and the speed of the computing machine. The processor produces an output data file which is read by a post-processor to view and evaluate results.

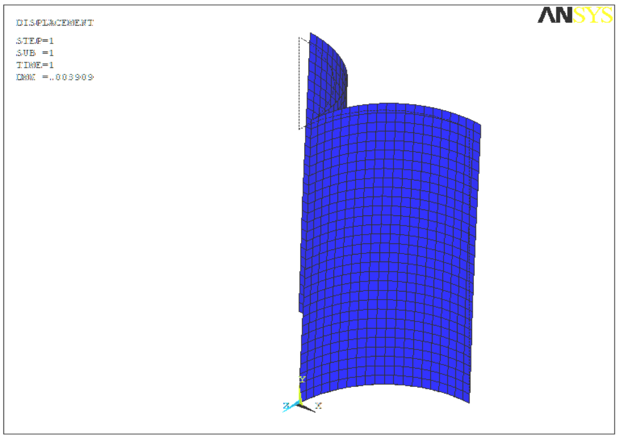

Figure 1. A shell modeled by shell elements [58].

10.1. Deriving element parameters

A finite element formulation begins with a formulation at the element level. The undetermined coefficients used in the previously described methods of Ritz and weighted residuals are replaced by unknown values (displacements) at the nodes. Algebraic polynomials that represent the displacements over the element are derived based upon for solid elements, the nodal displacements ui, vi and wi (where i is a typical node) constitute the DOF used at each node. For the least possible tetrahedron element that has four nodes, with 3 DOF per node, a total of 12 DOF will be needed per element. This yields a total of 5 DOF per node. For a triangular shell element, with three nodes, this results in at least 15 DOF over the element. Appropriate algebraic polynomials are then carefully selected for the u, v and w displacements over the domain of the element with the same number of undetermined coefficients. For shell elements that include shear deformation, five independent variables exist at ach boundary. This, in principle, added complexity to the problem at hand. Interestingly, the treatment of elements with shear deformation is easier than that of classical shell elements the reason is the condition of compatibility. For classical shell elements, compatibility requires both displacements and slopes to be continuous across the boundary of the elements (and not only at the nodal points). For classica1 elements, this requires a higher degree polynomial over the domain. For elements with shear deformation and with higher number of DOF per node, the polynomial needed is of lesser degree, and thus the formulation is simpler. A typical displacement function within the element can be represented by the following polynomial:

$$
\begin{aligned}
U(\varphi_1,\varphi_2,\varphi_3) = {} & \alpha_{000} + \alpha_{001}\varphi_3 + \alpha_{010}\varphi_2 + \alpha_{100}\varphi_1 + \alpha_{110}\varphi_1\varphi_2 + \alpha_{101}\varphi_1\varphi_3 \\
& + \alpha_{011}\varphi_2\varphi_3 + \alpha_{200}\varphi_1^2 + \alpha_{020}\varphi_2^2 + \alpha_{002}\varphi_3^2 + \alpha_{111}\varphi_1\varphi_2\varphi_3 \\
& + \alpha_{210}\varphi_1^2\varphi_2 + \alpha_{210}\varphi_1^2\varphi_3 + \alpha_{021}\varphi_1\varphi_2^2 + \alpha_{120}\varphi_1\varphi_3^2 + \alpha_{102}\varphi_1\varphi_3^2 \\
& + \alpha_{012}\varphi_2\varphi_3^2 + \alpha_{300}\varphi_1^3 + \alpha_{030}\varphi_2^3 + \alpha_{003}\varphi_3^3 + \alpha_{112}\varphi_1\varphi_2\varphi_3^2 \\
& + \alpha_{121}\varphi_1\varphi_2^2\varphi_3 + \alpha_{211}\varphi_1^2\varphi_2\varphi_3 + \ldots
\end{aligned}
\tag{9}
$$

The above expression is written carefully such that it begins with a constant term α_{000} followed by three linear terms. These four terms would be enough interpolation functions for a solid element. If a higher order element is selected, then quadratic terms must be included. Some or all of the quadratic terms described in the above equations can be selected for that purpose. Similar displacement functions can be written for other displacements like V and W (and shear functions for shear deformable shell elements). The coefficients in the polynomial are determined by using proper interpolation functions that achieve a displacement (or curvature) value at one of the nodes. The stiffness matrix for the element is determined by using either the Ritz method or one of the weighted residual methods. If the Ritz method is used, the trial functions of Eq. 9 are substituted into the Lagrangian expression, which in turn needs to be minimized using:

$$\frac{\partial L}{\partial \alpha_{ijk}} = 0 \sum_{j=l}^{n} K_{ij}^{e} u_j^{e} - F_i^{e} \tag{10}$$

The superscript (e) is used to refer to a particular element, the Kjj term refers to stiffness and inertia terms, uj refers to the DOF, and F j refers to the force vector at the prescribed DOF.

The stiffness matrices of laminated shell and plate elements exist in the literature and are available in many books including those of Reddy (1984, 2003). Coordinate transformation is needed to write the above equations in the global coordinates used in the total system, which are typically different than those used at the element level.

10.2. Assembly of the elements

Once the stiffness (and mass) matrix as well as the force vector are written for each element in terms of global coordinates, the elements need to be connected together to obtain one global stiffness matrix and another mass matrix as well as a force vector. Each node will have a global displacement in a certain direction. This displacement must be the same for each of the surrounding elements at that node; a total energy expression can be reached by summing all the potential energies of the elements:

$$\Pi = \sum_{i=l}^{N} \Pi_i \tag{11}$$

Where N is the total number of elements the variation of the total energy is then taken to 0, satisfying the principle of minimum potential energy for the whole system.

$$\frac{\partial \Pi}{\partial U_1} = 0, \qquad I = 1,2,\ldots,n \tag{12}$$

The above expression yields equations for each DOF for the global system. This, when connected together, can be written as a global matrix of the form:

$$K_{ij}u_i = F_i \qquad (13)$$

Note that the Kij terms include both stiffness and mass coefficients needed ill a dynamic analysis. The resulting stiffness and mass matrices are symmetric.

10.3. Imposing the boundary conditions and post-processing

The boundary conditions need to be imposed on the global equations. The specified boundaries can either be written in terms of specified displacements or specified forces (at any of the nodes). If the displacement is specified at a certain node, then the row and column corresponding to the node should be deleted. The force column should be modified by subtracting the product of the stiffness coefficient in the column with the specified displacement. For applied forces at any node, the internal forces should be equal to the external forces at that node (equilibrium at the node). Once the system is fully determined with only the displacements as unknown, typical mathematical procedures can be applied to obtain a solution (find the displacements at each of the nodes). Forces can then be computed at each of the boundaries well as the nodes.

10.4. Recent literature

Various finite elements for composite shells were discussed by Carrera (2001, 2002). Thin shell elements were used to analyze the vibrations of composite she1ls by many researchers. Thin and thick shallow shell elements were developed and used for composite shell vibration research in various studies. Finite elements were also developed using various, shear deformation theories. FEM were used to analyze stiffened shells by many authors. Predictor-corrector procedures, for stress and free vibration analyses of multilayered composite plates and shells were performed by Noor et al. (1990). Sanders' shell theory was modified to include, shear deformation and used to develop conforming finite elements by Farsakh and Qatu (1995). Thick conical shells were, analyzed using finite elements by Sivadas (1995). Hybrid-mixed formulations were used to study vibrations of composite shells. A higher-order, partial hybrid stress, FEM formulation was given by Yong and Cho (1995). 3D elements were also used in the analysis of composite shells by Dasgupta and Huang (1997). A discrete layer analysis using FEM was performed for cylindrical shells by Ramesh and Ganesan (1992, 1994) and Gautham and Ganesan (1992, 1994). Finite elements were developed for the new piezoelectric materials. FEM models were generated for shells with embedded layers of these materials by Jia and Rogers (1990). Other models were developed for shells made of such materials by many researchers. Shells made of shape memory alloys were treated by Hurlbut and Regelbrugge (1996). Berger and Gabbert (2000) used finite element analysis to design piezoelectric controlled smart structures. Nonlinear vibration analysis was performed by many researchers using various types of composite shell elements. Flutter was considered by Chowdary et al. (1994) and Liao and Sun (1993). Kapania and Byun (1992) considered Imperfections using finite element analysis.

11. CONCLUSION

The equations presented thus far are complete in the sense that the number of equations is equal to the number of unknowns for each of the theories presented.

The 3D elasticity theory requires the satisfaction of three boundary conditions at each boundary. 2D thin shell theories require the satisfaction of four conditions at each of the edges of the shell, along with an additional one at each of the free corners. The accuracy of this assumption depends heavily on how the unknowns are assumed to vary with the thickness. Generally this yields results closer to that of the 3D elasticity theory. It has also been shown here that accurate treatment of a first order expansion for shells yielded rather sophisticated stress resultant equations. Such equations, once derived properly, may indeed yield results closer to those obtained using the 3D elasticity equations. As stated earlier, such equations will introduce boundary terms that are not easily conceptualized. Other techniques exist for solving the vibration problem of composite shells including finite differences, boundary element methods (BEM), differential and others. Differential is evolving rapidly lately, and the same is true for BEM. Differential was used to analyze laminated conical shells by Shu (1996) and Wu (2000). Argyris and Tenek (1996) developed the "natural mode method" and used it in their analysis. Boundary domain elements were employed to analyze free vibration of thick shells by Wang and Schweizerhof (1996, 1997) and Beskos (1997). A Spline strip method was used to analyze thick cylindrical shells by Mizusawa and Kito (1995) and Mizusawa (1996). Boundary continuous displacement based Fourier analysis was conducted for laminated panels using a classical shallow shell theory by Chaudhunri and Kabir (1992). Li and Mirza (1997) used a method based on the superposition and state-space techniques to study the free vibration of cross-ply laminated composite shell panels with general boundary conditions. The relative ease, accuracy and dependability of the Ritz method for linear analysis and the Galerkin method for nonlinear analysis have proved to be valuable for simple structures and/or boundary conditions.

The FEM have also been widely used and accepted, for complex structures and boundary configuration. The availability of commercial codes, in obtaining natural frequencies using the FEM and the user friendliness of these codes have made finite elements the industry standard in many applications that involve dynamic analysis of composite structures.

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