J. Civil Eng. Mater.App. 2023 (March); 7(1): 11-21

Journal of Civil Engineering and Materials Application
Journal home page: http://jcema.com

Received: 22 January 2023 • Accepted: 28 February 2023

Research

doi: 10.22034/jcema.2023.382603.1101

# A Novel Analytical Approach for Assessing the Buckling Behavior of non-Prismatic Elastic Columns Based on Power Series

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## ABSTRACT

The analysis of the post-buckling behavior of elastic structures still needs the resolution of a body of non-linear differential equations according to equilibrium equations. Also, the yielding and buckling factors are important in the procedure of designing members under forces such as axial or conjoint axial force and bending moment. In such a way that if the length of the member is too long or the member is thin, before the yielding, buckling will occur in the member, and it is necessary to check and control the member for possible buckling. The present work deals with the stability analysis of elastic columns with variable cross-sections under concentrated end load and proposes a simplified approach to the evaluation of the critical buckling force of columns according to the assumptions of the Elastica theory. In this paper, the power series are used to simplify the equations. The numerical issues of the critical buckling force are presented for prismatic and non-prismatic columns subjected to end force, and the effectiveness of this approach is verified for buckling analysis of tapered columns, and the rate of accuracy is assessed. The elastic buckling force of elastic structures shows that the introduced model is computationally extremely efficient with the details presented in general. This paper should be a basic reference to compare the results with other research.

**Keywords:** Buckling force, Elastica theory, Non-prismatic column, Power series, Stability



## 1. INTRODUCTION

Stability analysis and determination of critical buckling force of columns has a very long history in structural engineering research. Frame members, including beams or columns, can be designed under different conditions and for different applications as variable cross-sections, and these members are called non-prismatic. A member with a variable cross-section has a higher bearing capacity than a prismatic member with a larger cross-section. Also, most engineers today are looking to use ideal methods to reduce the weight of the structure. As a result, it can be said that by using this type of member, the bearing capacity can be significantly increased, and at the same time, the weight of the structure can be reduced. This is very important in terms of economic savings.

Researchers have used various methods, most of which are based on solving the differential equation of stability of a general element under the influence of axial and lateral forces, to specify the buckling force of columns. In this regard, Timoshenko and Gere [1], Chen [2], and Bazant [3] proposed several methods based on approximate numerical methods or solving a body of differential equations governing and controlling the stability of elastic columns with different boundary conditions. Frisch-Fay [4] studied the buckling issue and specified the buckling force of a prismatic element with uniform axial force in different boundary conditions using the analytical method and calculated the critical load for a fully restrained column using the Bessel integral. Karabalis [5] solved the stability differential equation controlling the non-prismatic beam by the Numerical method and then determined the stiffness and geometric matrix coefficients governing the elastic member. During a numerical study, Lake [6] determined the buckling force of beam columns whose cross-sectional area changes in steps along the length of the member and is under static loading. Finally, it was well shown by several diagrams to what extent reducing the cross-sectional area along the member can be economical in terms of reducing the weight of the structure and also increasing the axial buckling force due Finally, it was well shown by several diagrams to what extent reducing the cross-sectional area along the member can be economical in terms of reducing the weight of the structure and also increasing the axial buckling force due to the increase in bearing capacity. Arbabi [7] examined the elastic buckling of non-prismatic beam columns with variable thickness. Williams and Aston [8] examined the limits of the buckling force of tapered columns with the second moment of area. With the aid of the Bessel functions, Li [9] gave a variety of exact solutions for buckling of non-uniform columns subjected to the axial concentrated and distributed loading. Signer [10] studied the buckling behavior of tapered columns with linear changes in flexural stiffness along the member. Then, the slope-deflection method is used for calculating the effective buckling length of non-prismatic columns Ermopulos [11]. The amount of critical buckling force of a column with a variable cross-section, which is a member of an unbraced frame and is subjected to concentrated axial compressive load at its various points, has been determined by Ermopulos [12] by solving the nonlinear differential equation of stability using the slope-deflection method. Raftoyiannis [13] examined the critical buckling force of muscular members. In this study, in order to solve the quadratic differential equation, it is assumed that the effect of the above can be considered as an initial and hypothetical curvature. Saffari [14] obtained the effective length coefficient for sloping frames with muscular members using the slope-deflection relationships. The proposed method modifies the stability coefficients in the slope-deflection relationships for frames whose profile dimensions change linearly along the member. Rahai [15] determined the critical loads of columns with variable cross-sections in different modes using the energy method and the principle of similarity of buckling and vibrating deformation of elastic members. The stability of a braced standing by a heavy column and its optimum place of support is evaluated by Wang [16]. The buckling of axially graded columns with any axial nonhomogeneity and the design of the shape profile of a column are assessed by Huang and Li [17]. Rokhi Shahri et al. [18] studied the post-buckling behavior of the lateral unbraced frame by using Elastica theory.

Many approaches are used to solve limitations to investigate the stability issues of elastic columns with changeable cross-sections subjected to different boundary conditions. The use of the special capability technique, for example, utilizing Bessel functions, emphatically relies upon the type of a customary differential condition with variable coefficients. This paper presents an accurate technique for deciding the buckling force of prismatic and non-prismatic columns under tip force.

## 2. METHODOLOGY

### 2.1. CLAMPED-FREE COLUMN

A prismatic column in which the flexural rigidity is variable is considered. The column under the axial compressive force P is shown in Fig. 1(a).The column length is L and clamped to the support at point A. In case the load is $P \geq P_{cr}$, where $P_{cr}$ is the critical load of column, the column can be positioned in the equilibrium state as shown in Fig.1(b). By accepting the simplifying assumption, the axial length change of the column is ignored.

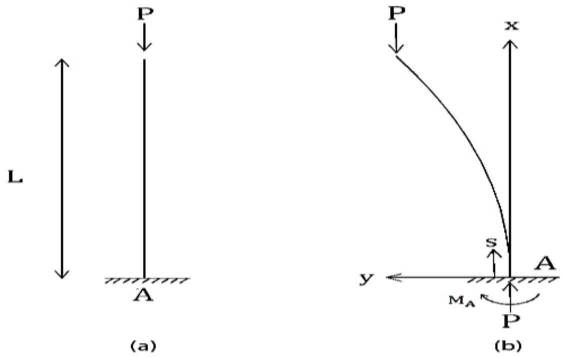


**Figure 1.** (a) A prismatic C-F column under the load P; (b) Equilibrium state of column.

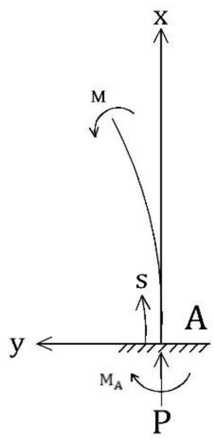


**Figure 2.** *The static equilibrium state of C-F column by drawing a cross section*

The boundary conditions in this structure are:

$$y(s=0)=0\,,\ \ \theta(s=0)=0\,,\ \ M(s=L)=0 \qquad (1)$$

Where $y$ is the horizontal displacement, $\theta$ is the slope value and $M$ is the internal bending moment.

The McLaurin expansion of the slope function, $\theta(s)$ is as follows:

$$\theta(s)=\sum_{n=0}^{M} a_n \frac{s^n}{n!} \qquad (2)$$

Where:

$$a_n=\frac{d^n\theta}{ds^n}, \qquad s=0 \qquad (3)$$

The zero slope in support A gives us the following relation:

$$a_0=0 \qquad (4)$$

The static equilibrium equations determine the distributed forces. By drawing a cross section in the structure, the static equilibrium state of column as in Figure 2 is available.

$$M = M_A - Py \tag{5}$$

On the other hand we know:

$$M = EI\frac{d\theta}{ds} \tag{6}$$

Where $EI$ is the flexural rigidity and a function of $s$. The combination of Eqs. (4) and (5) gives the following relation:

$$EI\frac{d\theta}{ds} = M_A - Py \tag{7}$$

For obtaining the McLaurin expansion coefficients of Eq. (3), Eq. (7) and its derivatives are used. Thus Eq. (8) results for $n = 1$.

$$EI_0 a_1 = M_A \tag{8}$$

Where $EI_0$ is the amount of flexural stiffness at the abutment. Deriving from Eq. (7) leads to the following equation:

$$\frac{d}{ds}\left[EI\frac{d\theta}{ds}\right] = -P\frac{dy}{ds} \tag{9}$$

Considering that

$$\frac{dy}{ds} = sin\ \theta \tag{10}$$

Eq. (9) may be written as:

$$\frac{d}{ds}\left[EI\frac{d\theta}{ds}\right] = -Psin\ \theta \tag{11}$$

The latter equation gives the following expression with respect to the boundary conditions and according to Eqs. (3) and (8).

$$EI_0' a_1 + EI_0 a_2 = 0 \tag{12}$$

or

$$a_2 = -\frac{EI_0'}{EI_0} a_1 \tag{13}$$

Coefficients $a_n (n > 2)$ can be determined by sequential derivation from the sides of Eq. (11).

$$\frac{d^n}{ds^n}\left[EI\frac{d\theta}{ds}\right] = -P\frac{d^{n-1}}{ds^{n-1}} Sin\ \theta \tag{14}$$

Given that the flexural stiffness is a function of $s$, the left side of the above equation is extended as follows:

$$\sum_{j=0}^{M}\binom{n}{j}\theta^{(j+1)}EI^{(n-j)} \qquad (15)$$

Where $\theta^{(j+1)}$ is the $(j+1)$th order derivative of the function $\theta(s)$ and $EI^{(n-j)}$ is the $(n-j)$th order derivative of the function $EI(s)$.

Using Eq. (3), the expression (15) can be rewritten as follows at the points $s = 0$:

$$EI_0 a_{n+1} + \sum_{k=1}^{M}\binom{n}{k-1} a_k EI_0^{(n+1-k)} \qquad (16)$$

The combination of Eq. (14) and expression (16) leads to

$$EI_0 a_{n+1} + \sum_{k=1}^{M}\binom{n}{k-1} a_k EI_0^{(n+1-k)} = -Pb_n \qquad (17)$$

or

$$a_{n+1} = -\frac{Pb_n}{EI_0} - \sum_{k=1}^{M}\binom{n}{k-1}\frac{EI_0^{(n+1-k)}}{EI_0}a_k \qquad (18)$$

The expression to the right of Eq. (14) is denoted by $b_n$ at point $s = 0$. The above equation can obtain all McLaurin expansion coefficients in terms of $a_1$.

The following boundary condition can obtain the coefficient $a_1$ in terms of the load $P$.

$$\frac{d\theta}{ds} = 0, \quad s = L \qquad (19)$$

Substitution of Eq. (2) into Eq. (19) leads to the following equation:

$$\sum_{n=1}^{M} a_n \frac{L^{n-1}}{(n-1)!} = 0 \qquad (20)$$

Considering the coefficients $a_n$ in terms of $a_1$ and $P$, the equation can be expressed as follows:

$$f(a_1, P) = 0 \qquad (21)$$

In case the load is $P > P_{cr}$, $a_1$ is non-zero and the larger the load is applied, the coefficient $a_1$, which is the slope rate at the support, increases. Therefore, the equation that can give the critical load is obtained by verging $a_1$ to zero. Solving Eq. (22) provides the critical load for the column.

$$f(0, P_{cr}) = 0 \qquad (22)$$

## 2. HINGED-HINGED COLUMN

In this state, the column under the axial compressive force P, is considered as shown in Figure 3. The boundary conditions are presented in the following:

$$y(s = 0) = 0 \qquad (23)$$

$$y(s = L) = 0 \qquad (24)$$

$$M(s=0)=0 \tag{25}$$

$$M(s=L)=0 \tag{26}$$

The slope at point A in Figure 4 is $\theta_0$ $(a_0=\theta_0)$ and$a_1=0$. Establishing static equilibrium equations for a part of the structure leads to the following Eqs:

$$M=-Py \tag{27}$$

$$-EI\frac{d\theta}{ds}=-Py \tag{28}$$

To calculate McLaurin expansion coefficients, we derive from the sides of Eq. (28).

$$\frac{d}{ds}\left[EI\frac{d\theta}{ds}\right]=P\sin\theta \tag{29}$$

Considering Eqs. (3) and (7) and using the Eq. (29) at $s=0$, the following Eq is reached.

$$EI_0'a_1+EI_0a_2=P\sin\theta_0 \tag{30}$$

Given that$a_1=0$, the coefficient $a_2$ is obtained from the above equation. The calculation of the coefficients $a_n$ for $n>2$ results by sequential derivation from the parties of Eq. (30).

$$EI_0a_{n+1}+\sum_{k=1}^{M}\binom{n}{k-1}a_kEI_0^{(n+1-k)}=Pb_n \tag{31}$$

Where

$$b_n=\frac{d^{n-1}}{ds^{n-1}}Sin\,\theta,\qquad s=0 \tag{32}$$

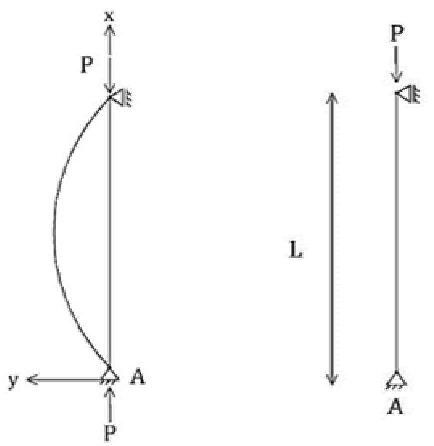


**Figure 1.** A prismatic H-H column under the load P

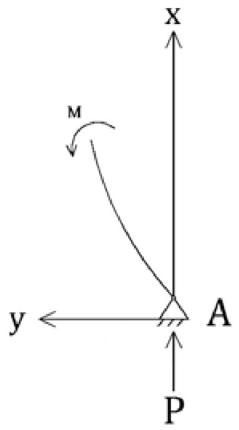


**Figure 2.** The static equilibrium state of H-H. Column

Using Eqs. (30) and (31), all McLaurin expansion coefficients are obtained in terms of $\theta_0$ and$P$. The following boundary condition is considered:

$$\frac{d\theta}{ds} = 0, \quad s = L \tag{33}$$

The Eq. (33) is extended as follows:

$$\sum_{n=1}^{M} a_n \frac{L^{n-1}}{(n-1)!} = 0 \tag{39}$$

The substitution of the coefficients $a_n$ in terms of $\theta_0$ and $P$ in the last equation provides the following equation.

$$g(P, \theta_0) = 0 \tag{40}$$

According to the concept of neutral equilibrium in structural stability in the limit state$\theta_0 \rightarrow 0$, the applied load $P$ tends to $P_{cr}$.

## 3. RESULTS AND DISCUSSION

In this section, the accuracy and precision of the proposed method is assessed. At first, the buckling behavior of a prismatic C-F column is studied and the results are compared with Timoshenko [1]. Then the non-prismatic columns will be examined. Equations are solved with the help of Mathematica [19].

### 3.1. EXAMPLE 1

In this example for a clamped-free column, $\frac{P}{P_{cr}} = \alpha$ and the column buckling is examined for different α. Also, the free horizontal and vertical displacement of the column is calculated based to the following equations using the McLaurin series.

$$y_L = \int_0^L Sin\, \theta\, ds = \int_0^L (\theta - \frac{\theta^3}{3!} + \cdots)\, ds \tag{41}$$

**Table 1.** Angle $\theta_l$ and displacement of free end of a prismatic C-F column for different α for M = 20.

| $\alpha$ | 1 | 1.015 | 1.063 | 1.152 | 1.293 |
|---|---|---|---|---|---|
| $\theta_l$ | 0 | 19.743° | 39.755° | 58.819° | 69.149° |
| $x_l/L$ | 1 | 0.970 | 0.883 | 0.750 | 0.655 |
| $y_l/L$ | 0 | 0.216 | 0.420 | 0.585 | 0.666 |

**Table 2.** Angle $\theta_l$ and displacement of free end of a prismatic C-F column for different α for M = 40.

| $\alpha$ | 1 | 1.015 | 1.063 | 1.152 | 1.293 |
|---|---|---|---|---|---|
| $\theta_l$ | 0 | 19.743° | 39.8° | 60.04° | 76.593° |
| $x_l/L$ | 1 | 0.970 | 0.882 | 0.741 | 0.592 |
| $y_l/L$ | 0 | 0.217 | 0.420 | 0.594 | 0.705 |

$$x_L = \int_0^L Cos\,\theta\, ds = \int_0^L \left(1 - \frac{\theta^2}{2!} + \cdots\right) ds \qquad (42)$$

The results are shown in Tables 1 and 2. As can be seen from the comparison of the Tables 1 and 2, a more accurate answer is obtained by increasing the number of sentences in the McLaurin series. The results are compared with Timoshenko's research [1] and are shown in Table 3.

**Table 3.** Comparison of current study with Timoshenko theory

| $\alpha$ | $\theta_l$ | | $x_l/L$ | | $y_l/L$ | |
|---|---|---|---|---|---|---|
| | **Proposed** | **Timoshenko** | **Proposed** | **Timoshenko** | **Proposed** | **Timoshenko** |
| **1** | 0 | 0 | 1 | 1 | 0 | 0 |
| **1.015** | 19.743° | 20 | 0.970 | 0.970 | 0.217 | 0.220 |
| **1.063** | 39.8° | 40 | 0.882 | 0.881 | 0.420 | 0.442 |
| **1.152** | 60.04° | 60 | 0.741 | 0.741 | 0.594 | 0.593 |
| **1.293** | 76.593° | 80 | 0.592 | 0.560 | 0.705 | 0.719 |

### 3.2. EXAMPLE 2

This example previously published [20] is presented, for which the proposed approach is demonstrated and the results are compared and validated.

As indicated in Figure 5, three non-prismatic columns are shown in three different boundary conditions (C-F, H-H and C-H). The cross section of all columns is rectangular, the width of which changes as a quadratic curve along the length of the member, and at the upper end is reduced to half of its initial value. In addition the columns are under axial compressive load P.

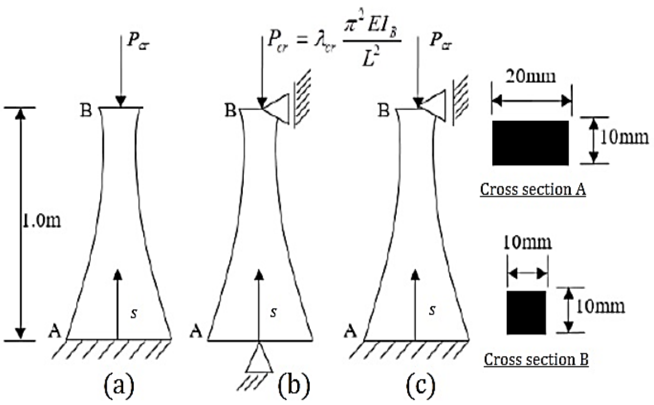


**Figure 5.** Non-prismatic columns with different boundary conditions under concentrated compressive load

The critical buckling force of the column is $P_{cr} = \lambda_{cr} \frac{\pi^2 EI_B}{L^2}$ where $\lambda_{cr}$the critical buckling force coefficient is. The Young's modulus is 25 GPa and the moment of inertia and cross-sectional area of the analyzed column in local coordinates are determined as follows:

$$I(s) = I_A(1 - 0.5s^2) \quad (43)$$

$$A(s) = A_A(1 - 0.5s^2) \quad (44)$$

The columns are shown in Figure 5 are analyzed by using the proposed approach by Wolfrom Mathematica software compared to those given by the Ref. [21]. Table 4 gives the critical buckling force coefficient values obtained for different methods.

**Table 4.** Critical buckling load coefficient $\lambda_{cr}$ for non-prismatic columns of Figure 5.

| Method | C-F | H-H | C-H |
|---|---|---|---|
| **FEM Ansys [22]** | 0.463 | 1.686 | 3.348 |
| **Ref. [21]** | 0.4629 | 1.6877 | 3.349 |
| **Present proposed study** | 0.4631 | 1.6877 | 3.591 |

3.3. EXAMPLE 3

Consider the non-prismatic elastic column under a compressive force P at the end $x = L$ in three different boundary conditions (C-F, H-H and C-H) as shown in Figure 6. The cross section of the column is rectangular, in

which its height decreases linearly while its width is constant. The moment of inertia of the column can be expressed as

$$I(\xi) = I_0(1 - \beta\xi)^3 \quad (45)$$

Where $\beta$ denotes the taper ratio of the column.

Numerical results of the critical buckling force $\lambda_P = \frac{PL^2}{EI_0}$ are calculated and they are illustrated in Table 5. These results agree very well with those given by the Ref. [21] and the finite-element method (FEM) [22].

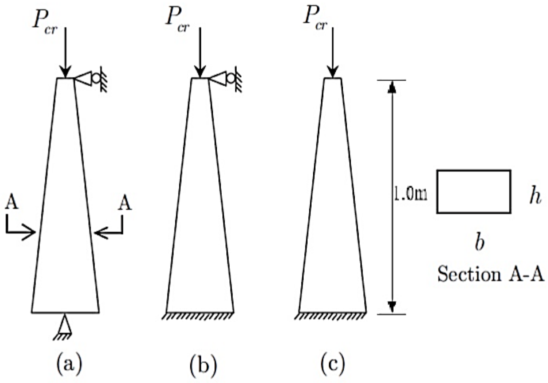


**Figure 6.** Geometry and boundary condition of axially loaded column: (a) hinged; (b) clamped hinged; and (c) clamped free

**Table 5.** Critical load $\lambda_P$ for a non-prismatic column in three different boundary conditions

| $\beta$ | C-F | | | H-H | | | C-H | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Proposed** | **Ref. [21]** | **FEM [22]** | **Proposed** | **Ref. [21]** | **FEM [22]** | **Proposed** | **Ref. [21]** | **FEM [22]** |
| 0 | 2.467 | 2.467 | 2.47 | 9.870 | 9.870 | 9.87 | 20.216 | 20.191 | 20.19 |
| 0.2 | 2.023 | 2.023 | 2.02 | 7.091 | 7.090 | 7.09 | 14.445 | 14.494 | 14.50 |
| 0.4 | 1.569 | 1.569 | 1.57 | 4.685 | 4.685 | 4.69 | 9.929 | 9.543 | 9.55 |
| 0.6 | 1.098 | 1.098 | 1.10 | 2.672 | 2.672 | 2.68 | 4.746 | 5.388 | 5.40 |
| 0.8 | 0.606 | 0.597 | 0.60 | 1.088 | 1.082 | 1.09 | 1.790 | 2.119 | 2.13 |

## 4. CONCLUSION

The presented examples and comparison of the results of this paper with reference indicated a good correlation and showed the proposed method can be introduced as a suitable method for evaluating the post-buckling behavior of columns and specifying the buckling force of non-prismatic columns. Obviously, in order to achieve a more accurate response requires using more sentences in the McLaurin series; however, software limitations in some cases may prevent this aim.

FUNDING/SUPPORT

Not mentioned any Funding/Support by authors.

ACKNOWLEDGMENT

Not mentioned by authors.

AUTHORS CONTRIBUTION

This work was carried out in collaboration among all authors.

CONFLICT OF INTEREST

The author (s) declared no potential conflicts of interests with respect to the authorship and/or publication of this paper.

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